AN ANALYTIC DERIVATION

OF THE EFFICIENT PORTFOLIO FRONTIER

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I. Introduction. The characteristics of the efficient (in the mean-variance sense) portfolio frontier have been discussed at length in the literature.[1] However, for more than three assets, the general approach has been to display qualitative results in terms of graphs.[2] In this paper, the efficient portfolio frontiers are derived explicitly, and the characteristics claimed for these frontiers verified. The most important implication derived from these characteristics, the separation theorem, is stated and proved in the context of a mutual fund theorem. It is shown that under certain conditions, the classical graphical technique for deriving the efficient portfolio frontier is incorrect.

II. The efficient portfolio set when all securities are risky. Suppose there are m risky securities with the expected return on the ith security denoted by α_i; the covariance of returns between the ith and j^{th} security denoted by σ_{ij}; the variance of the return on the ith security denoted by $\sigma_{ii} = \sigma_i^2$. Because all m securities are assumed

[1]See H. Markowitz [6], J. Tobin [9], W. Sharpe [8], and E. Fama [3].

[2]The exceptions to this have been discussions of general equilibrium models: see, for example, E. Fama [3], F. Black [1], and J. Lintner [5].

risky, then $\sigma_i^2 > 0$, i = 1, . . ., m, and we further assume that no security can be represented as a linear combination of the other securities, i.e. the variance-covariance matrix of returns, $\Omega = [\sigma_{ij}]$, is non-singular. The frontier of all feasible portfolios which can be constructed from these m securities is defined as the locus of feasible portfolios which have the smallest variance for a prescribed expected return. Let δ_i = percentage of the value of a portfolio invested in the ith security, i = 1, . . ., m, and as a definitional result, $\sum_1^m \delta_i = 1$. Then, the frontier can be described as the set of portfolios which satisfy the constrained minimization problem,

(1)
$$\min \frac{1}{2} \sigma^2$$

subject to
$$\sigma^2 = \sum_1^m \sum_1^m \delta_i \delta_j \sigma_{ij}$$

$$\alpha = \sum_1^m \delta_i \alpha_i$$

$$1 = \sum_1^m \delta_i,$$

where σ^2 is the variance of the portfolio on the frontier with expected return equal to α.[3] Using Lagrange multipliers,

(1) can be re-written as,

(2) $\min \left\{ \frac{1}{2} \sum_1^m \sum_1^m \delta_i \delta_j \sigma_{ij} + \lambda_1 [\alpha - \sum_1^m \delta_i \alpha_i] + \lambda_2 [1 - \sum_1^m \delta_i] \right\}$

$\{\delta_1, \cdot \cdot \cdot, \delta_m, \lambda_1, \lambda_2\}$

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[3]The only constraint on the δ_i is that they sum to unity, and hence borrowing and short-selling of all securities is allowed. Obviously, the minimization of $\frac{1}{2}\sigma^2$ will minimize σ^2.

where λ_1 and λ_2 are the multipliers. A critical point occurs where the partial derivatives of (2) with respect to $\delta_1, \ldots, \delta_m, \lambda_1,$ and λ_2 are equal to zero, i.e.

(3a)
$$0 = \sum_1^m \delta_j \sigma_{ij} - \lambda_1 \alpha_i - \lambda_2, \quad i = 1, \ldots, m$$

(3b)
$$0 = \alpha - \sum_1^m \delta_i \alpha_i$$

(3c)
$$0 = 1 - \sum_1^m \delta_i .$$

Further, the δ's which satisfy (3) minimize σ^2 and are unique by the assumption on Ω. System (3) in linear in the δ's and hence, we have from (3a) that

(4)
$$\delta_k = \lambda_1 \sum_1^m v_{kj} \alpha_j + \lambda_2 \sum_1^m v_{kj}, \quad k=1, \ldots, m,$$

where the v_{ij} are defined as the elements of the inverse of the variance-covariance matrix, i.e. $\Omega^{-1} \equiv [v_{ij}]$. Multiplying (4) by α_k and summing over $k=1, \ldots, m$, we have that

(5)
$$\sum_1^m \delta_k \alpha_k = \lambda_1 \sum_1^m \sum_1^m v_{kj} \alpha_j \alpha_k + \lambda_2 \sum_1^m \sum_1^m v_{kj} \alpha_k,$$

and by summing (4) over $k = 1, \ldots, m$, we have that

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[4] Ω is a non-singular variance-covariance matrix, and therefore, symmetric and positive definite. It follows directly that Ω^{-1} is also. Hence $v_{kj} = v_{jk}$ for all j and k, and B and C are quadratic forms of Ω^{-1} which means that they are strictly positive (unless all $\alpha_i = 0$).

$$(6) \qquad \sum_1^m S_k = \lambda_1 \sum_1^m \sum_1^m v_{kj} \alpha_j + \lambda_2 \sum_1^m \sum_1^m v_{kj}.$$

Define: $\quad A \equiv \sum_1^m \sum_1^m v_{kj} \alpha_j$; $\qquad B \equiv \sum_1^m \sum_1^m v_{kj} \alpha_j \alpha_k$;

$$C \equiv \sum_1^m \sum_1^m v_{kj}.$$

From (3b), (3c), (5), and (6), we have a simple linear system for λ_1 and λ_2,

$$(7) \qquad \alpha = B\lambda_1 + A\lambda_2$$

$$1 = A\lambda_1 + C\lambda_2$$

where we note that $\sum_1^m \sum_1^m v_{kj} \alpha_j = \sum_1^m \sum_1^m v_{kj} \alpha_k$ and that $B > 0$ and $C > 0$.[4] Solving (7) for λ_1 and λ_2, we find that

$$(8) \qquad \lambda_1 = \frac{(C\alpha - A)}{D}$$

$$\lambda_2 = \frac{(B - A\alpha)}{D}$$

where $D \equiv BC - A^2 > 0$.[5] We can now substitute for λ_1 and λ_2 from (8) into (4) to solve for the proportions of each risky asset held in the frontier portfolio with expected return α: namely,

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[5]Because Ω^{-1} is positive definite, $0 < \sum_1^m \sum_1^m v_{ij} (B\alpha_i - A)(B\alpha_j - A)$
$= B^2 C - 2A^2 B + A^2 B = B(BC - A^2) = BD$. But $B > 0$, hence $D > 0$.

(9) $\quad \delta_k = \dfrac{\alpha \sum_1'^m v_{kj}(C\alpha_j - A) + \sum_1^m v_{kj}(B - A\alpha_j)}{D} \quad , \; k = 1, \ldots, m.$

Multiply (3a) by δ_i and sum from $i = 1, \ldots, m$ to derive

(10) $\qquad \sum_1^m \sum_1^m \delta_i \delta_j \sigma_{ij} = \lambda_1 \sum_1^m \delta_i \alpha_i + \lambda_2 \sum_1^m \delta_i.$

From the definition of σ^2, (3b), and (3c), (10) implies

(11) $\qquad \sigma^2 = \lambda_1 \alpha + \lambda_2.$

Substituting for λ_1 and λ_2 from (8) into (11), we write the equation for the variance of a frontier portfolio as a function of its expected return, as

(12) $\qquad \sigma^2 = \dfrac{C\alpha^2 - 2A\alpha + B}{D}$

Thus, the frontier in mean-variance space is a parabola. Examination of the first and second derivatives of (12) with respect to α shows that σ^2 is a strictly convex function of α with a unique minimum point where $\dfrac{d\sigma^2}{d\alpha} = 0$, i.e.

(13) $\qquad \dfrac{d\sigma^2}{d\alpha} = \dfrac{2[C\alpha - A]}{D}$

$\qquad\qquad\qquad = 0 \quad \text{when} \quad \alpha = \dfrac{A}{C}$

$\qquad \dfrac{d^2\sigma^2}{d\alpha^2} = \dfrac{2C}{D} > 0.$

Figure 1 is a graph of (12) where $\bar{\alpha} \equiv$ A/C and $\bar{\sigma}^2 \equiv$ 1/C are the expected return and variance of the <u>minimum-variance portfolio.</u> Define $\bar{\delta}_k$ to be the proportion of the minimum-variance portfolio invested in the kth asset, then from (9),

$$(14) \qquad \bar{\delta}_k = \frac{\sum_1^m v_{kj}}{C}, \quad k = 1, \ldots, m.$$

It is usual to present the frontier in the mean-standard deviation plane instead of the mean-variance plane. From (12) and (13), we have that

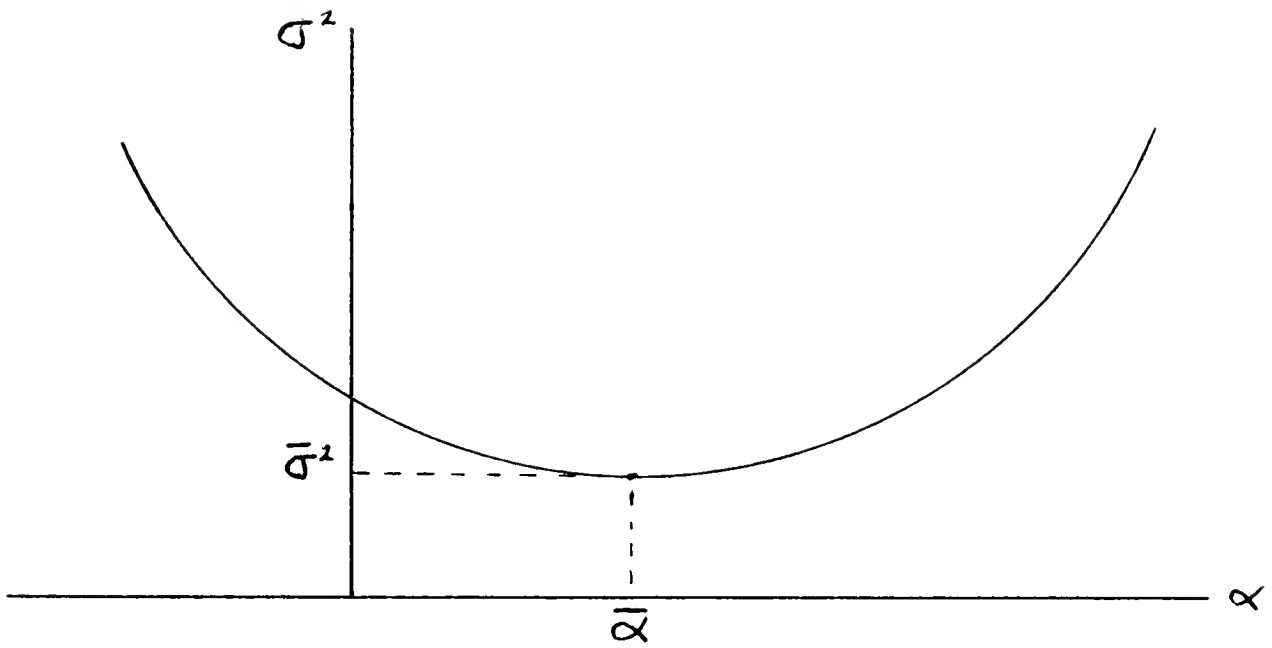$$(15) \qquad \sigma = \sqrt{(C\alpha^2 - 2A\alpha + B)/D}$$

$$\frac{d\sigma}{d\alpha} = \frac{(C\alpha - A)}{D\sigma}$$

$$\frac{d^2\sigma}{d\alpha^2} = \frac{1}{D\sigma^3} > 0.$$

From (15), σ is a strictly convex function of α, and the minimum standard deviation portfolio is the same as the minimum-variance portfolio. Figure 2 graphs the frontier in the standard form with α on the ordinate and σ on the abscissa. The broken lines are the asymptotes of the frontier whose equations are

$$(16) \qquad \alpha = \bar{\alpha} \pm \sqrt{\frac{D}{C}}\sigma.$$

Figure 1.

The efficient portfolio frontier (the set of feasible portfolios which have the largest expected return for a given standard deviation) is the heavy-lined part of the frontier in Figure 2, starting with the minimum-variance portfolio and moving to the North-East. The equation for α as a function of σ along the frontier is

$$(17) \qquad \alpha = \frac{A}{C} \pm \frac{1}{C} \sqrt{D(C\sigma^2 - 1)}$$

$$= \bar{\alpha} \pm \frac{1}{C} \sqrt{D(C\sigma^2 - 1)}$$

The equation for the efficient portfolio frontier is

$$(18) \qquad \alpha = \bar{\alpha} + \frac{1}{C} \sqrt{DC(\sigma^2 - \bar{\sigma}^2)} \qquad .$$

III. A mutual fund theorem.[6]

Theorem I. Given m assets satisfying the conditions of Section II, there exist two portfolios ("mutual funds") constructed from these m assets, such that all risk-averse individuals, who choose their portfolios so as to maximize utility functions dependent only on the mean and variance of their portfolios, will be indifferent between choosing portfolios from among the original m assets or from these two funds.

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[6]For a general discussion of mutual fund or "Separation" theorems, see D. Cass and J. Stiglitz [2]. In a theorem, in R. Merton [7], similar to the one in this section, it was incorrectly claimed that the two funds were unique. In fact, they are unique only up to a non-singular transformation.

Figure 2.
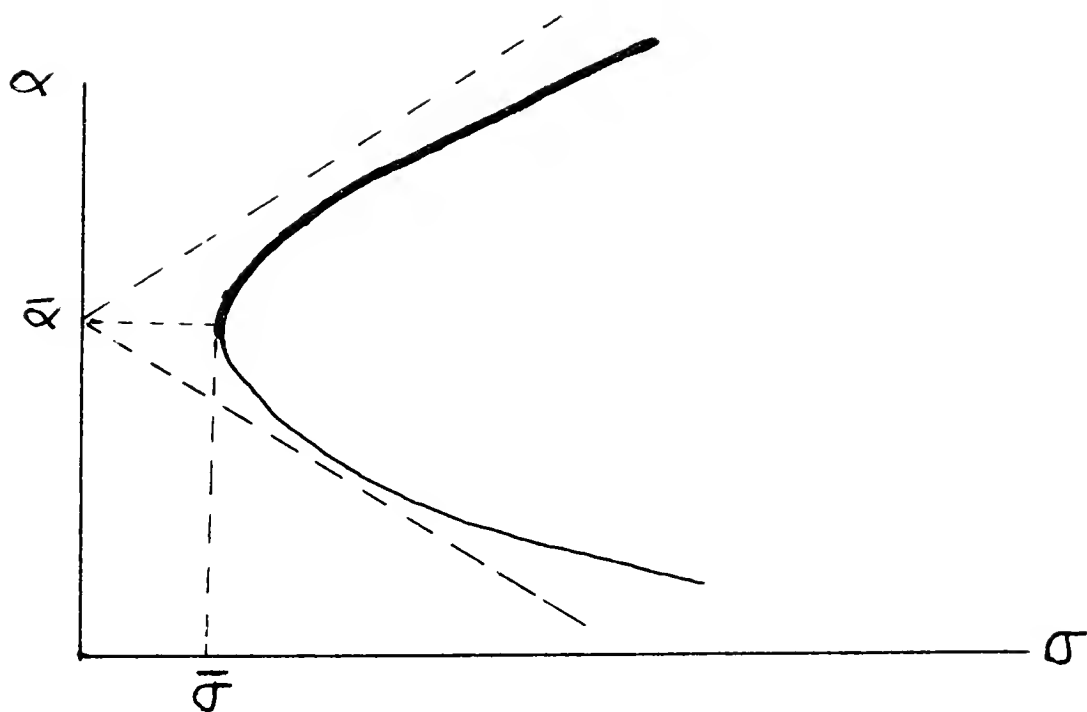
To prove theorem I. it is sufficient to show that any portfolio on
the efficient frontier can be attained by a linear combination of two
specific portfolios because an optimal portfolio for any individual
(as described in the theorem) will be an efficient portfolio.

Equation (9) describes the proportion of the frontier portfolio,
with expected return α, invested in the kth asset, $k = 1, \ldots, m$.
If we define

(19)
$$g_k = \sum\nolimits_1^m v_{kj}(C\alpha_j - A)/D, \quad k = 1, \ldots, m$$

$$h_k = \sum\nolimits_1^m v_{kj}(B - A\alpha_j)/D, \quad k = 1, \ldots, m,$$

then (9) can be re-written compactly as

(20)
$$\delta_k' = \alpha g_k + h_k, \quad k = 1, \ldots, m.$$

Note that, by their definitions, $\sum\nolimits_1^m g_k = 0$ and $\sum\nolimits_1^m h_k = 1$.

Because we want all individuals to be able to construct their
optimal portfolios from just two funds, the proportions of risky assets
held by each fund must be independent of preferences (or equivalently,
independent of α). Let a_k be the proportion of the first fund's
value invested in the kth asset, and let b_k be the proportion of the
second fund's value invested in the kth asset ($\sum\nolimits_1^m a_k = \sum\nolimits_1^m b_k = 1$,
and a_k and b_k must satisfy

(21)
$$\delta_k = \alpha g_k + h_k = \lambda a_k + (1 - \lambda)b_k, \quad k = 1, \ldots, m,$$

where λ is the particular "mix" of the funds which generates the efficient portfolio with expected return α.[7] All solutions to (21) will have $\lambda = \nu \alpha - \eta$ where ν and η are constants ($\nu \neq 0$) which depend on the expected returns of the two funds, α_a and α_b, respectively. Substituting for λ in (21) and imposing the condition that a_k and b_k be independent of α, we have that a_k and b_k must satisfy

(22)
$$g_k = \nu (a_k - b_k)$$

$$h_k = b_k - \eta (a_k - b_k), \quad k = 1, \ldots, m$$

For $\nu \neq 0$, (22) can solved for a_k and b_k to give

(23)
$$a_k = b_k + g_k / \nu$$

$$b_k = h_k + \eta g_k / \nu, \quad k = 1, \ldots, m.$$

a and b are two linearly independent vectors which form a basis for the vector space of frontier portfolios, δ.[8] Two portfolios whose holdings satisfy (23) will be called a set of basis portfolios. Two such portfolios must be frontier portfolios although they need not be efficient. Hence, from (20), both funds holdings are completely determined by their expected returns. Because $\alpha_a \equiv \sum_1'^m a_k \alpha_k$, $\alpha_b \equiv \sum_1'^m b_k \alpha_k$, $\sum_1'^m h_k \alpha_k = 0$, and $\sum_1'^m g_k \alpha_k = 1$, we have that

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[7]Two funds with proportions a_k and b_k which satisfy (21) will generate all frontier portfolios, including as a subset, the efficient portfolios.

[8]a and b are the m-vectors with elements a_k and b_k, $k = 1, \ldots, m$. δ is a m-vector with elements δ_k, where the δ_k satisfy (20).

(24)
$$\alpha_a = \frac{(1 + \eta)}{\nu}$$

$$\alpha_b = \frac{\eta}{\nu} .$$

Alternatively, given values for α_a and α_b, the constants ν and η can be written as

(25)
$$\nu = \frac{1}{(\alpha_a - \alpha_b)}$$

$$\eta = \frac{\alpha_b}{(\alpha_a - \alpha_b)} .$$

Different values for ν and η correspond to non-singular transformations of one basis set of portfolios into another basis, and as can be seen in (25), their values are independent of preferences. Thus, the investor need only know the means, variances, and covariances of the two funds to determine the mix, λ, which generate his optimal portfolio. The funds' "managers" can choose ν and η arbitrarily ($\nu \neq 0$) and then, follow the investment program prescribed in (23) without knowledge of individual preferences or wealth distribution. Hence, Theorem I is proved.

The essential characteristics of a set of basis portfolios are the expected returns, variances, and covariances. Equation (24) describes how the expected returns depend on ν and η. Because both portfolios are frontier portfolios, (24) and (12) can be combined to determine the variance of the first fund, σ_a^2, and of the second fund, σ_b^2, to be,

(26) $$\sigma_b^2 = (C\eta^2 - 2A\eta\nu + B\nu^2)/D\nu^2$$

and

(27) $$\sigma_a^2 = \sigma_b^2 + [C + 2(\eta C - A\nu)]/D\nu^2.$$

To find the covariance, σ_{ab}, we use (23) as follows:

(28) $$\sigma_{ab} = \sum_1^m \sum_1^m a_i b_j \sigma_{ij}$$

$$= \sum_1^m \sum_1^m b_i b_j \sigma_{ij} + \frac{1}{\nu}\sum_1^m \sum_1^m g_i n_j \sigma_{ij} + \frac{\eta}{\nu^2} \sum_1^m \sum_1^m g_i g_j \sigma_{ij}$$

$$= \sigma_b^2 - [\frac{A\nu}{C} - \eta]\,(\frac{C}{D\nu^2}).$$

Using (26) and (28), we can find those combinations of ν and η which will make the two portfolios uncorrelated (i.e. $\sigma_{ab} = 0$). For $\nu \neq 0$; $\sigma_{ab} = 0$ when

(29) $$C\eta^2 + B\nu^2 - 2A\eta\nu + C\eta - A\nu = 0.$$

(29) is an equation for a conic section, and because $A^2 - BC = -D < 0$, it must be an equation for an ellipse (See Figure 3.).

If we restrict both portfolios to be efficient[9] and take the convention that $\sigma_a^2 > \sigma_b^2$, then $\alpha_a > \alpha_b \geq \tilde{\alpha} = A/C$, and from (25), ν must be positive and $\eta \geq A\nu/C$. One could show that the line

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[9]Although the paper does not impose general equilibrium market clearing conditions, it is misleading to allow as one of the mutual funds a portfolio which no investor would ever hold long.

Figure 3.
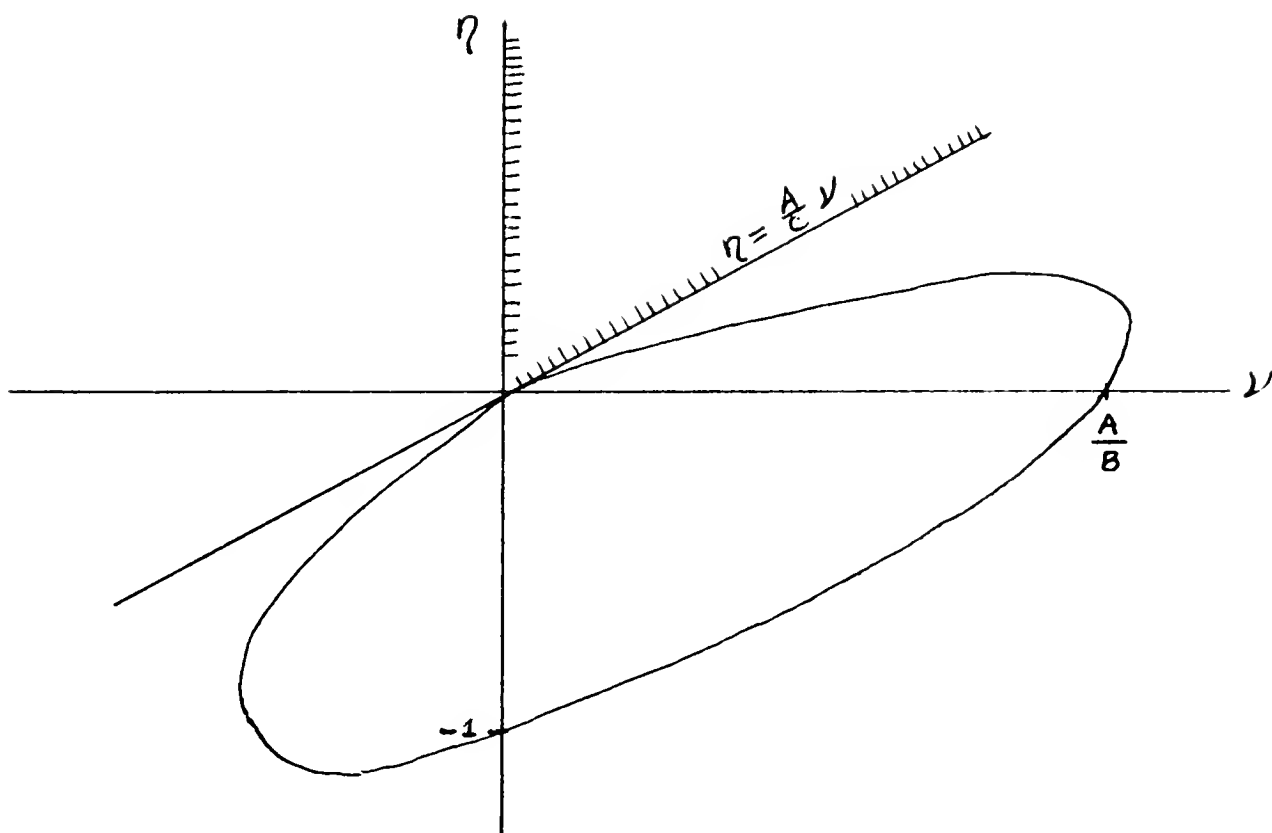
$\eta = A\nu/C$ is tangent to the ellipse at the point ($\eta = 0$, $\nu = 0$) as drawn in Figure 3. Therefore, there do not exist two efficient port- folios which are uncorrelated. From (28), we have that $\sigma_{ab} \geq \sigma_b^2 > 0$, which implies that all efficient portfolios are positively correlated. Further, $\sigma_{ab}^2 = \sigma_b^2$ if and only if $\eta = A\nu/C$. If $\eta = A\nu/C$, then, from (24), $\alpha_b = A/C = \bar{\alpha}$ which implies that the portfolio held by the fund with proportions b is the minimum-variance portfolio with $\sigma_b^2 = 1/C$ and $b_k = \bar{S}_k$. Because $1/C$ is the smallest variance of any feasible portfolio, it must be that, for efficient portfolios, σ_{ab} will be smallest when one of the portfolios is the minimum-variance portfolio. In this case, the portfolio of the other fund will have the character- istics that

(30)
$$\alpha_a = \frac{1}{\nu} + \frac{A}{C}$$

$$\sigma_a^2 = \frac{1}{C} + \frac{C}{D\nu^2}$$

$$a_k = \frac{\sum_1^m v_{kj}}{C} + \frac{g_k}{\nu} , \quad k = 1 \ldots, m,$$

where ν is arbitrary.

There does not appear to be a "natural" choice for the value of ν. However, it will be useful to know the characteristics of the frontier portfolio which satisfies

(31)
$$\frac{d\alpha}{d\sigma} = \frac{\alpha - R}{\sigma}$$

for some given value of R. From (15), $\frac{d\alpha}{d\sigma}$ along the frontier equals $D\sigma/(C\alpha - A)$. If we choose ν such that the portfolio with proportions

a satisfy (30), then

(32a)
$$\nu = \frac{C(A - CR)}{D}$$

(32b)
$$a_k = \frac{\sum_{1}^{m} v_{kj}(\alpha_j - R)}{(A - RC)} , \quad k = 1, \ldots, m.$$

If $R < \bar{\alpha} = A/C$, then $\nu > 0$ and the portfolio is efficient. If $R > \bar{\alpha}$, then $\nu < 0$, and the portfolio will be inefficient. If $R = \bar{\alpha}$, $\nu = 0$, and equation (31) cannot be satisfied by <u>any</u> frontier portfolio with finite values of α and σ. The implications of these results will be discussed in the following section.

IV. <u>The efficient portfolio set when one of the assets is risk-less.</u> The previous sections analyzed the case when all the available assets are risky. In this section, we extend the analysis to include a risk-less asset, by keeping the same m risky assets as before and adding a $(m+1)^{st}$ asset with a guaranteed return R. In an analogous way to (2) in Section II, the frontier of all feasible portfolios is determined by solving the problem:

(33)
$$\min \left\{ \frac{1}{2} \sum_{1}^{m} \sum_{1}^{m} \delta_i \delta_j \sigma_{ij} + \lambda [\alpha - R - \sum_{1}^{m} \delta_i (\alpha_i - R)] \right\}.$$

Notice that the constraint $\sum_{1}^{m+1} \delta_i = 1$ does not appear in (33) because we have explicitly substituted for $\delta_{m+1} = 1 - \sum_{1}^{m} \delta_i$, i.e. the $\delta_1, \ldots, \delta_m$ are unconstrained by virtue of the fact that δ_{m+1} can always be chosen such that $\sum_{1}^{m+1} \delta_i = 1$ is satisfied. This substitution not only simplifies the analytics of solving (33), but also will

provide insight into some results derived later in the paper.

The first-order conditions derived from (33) are

(34a) $\qquad 0 = \sum_1^m \delta_j \sigma_{ij} - \lambda(\alpha_i - R), \quad i = 1, \ldots, m$

(34b) $\qquad 0 = \alpha - R - \sum_1^m \delta_i(\alpha_i - R).$

Clearly, if $\alpha = R$, the frontier portfolio is $\delta_i = 0$, $i = 1, \ldots, m$
and $\lambda = 0$. When $\alpha \neq R$, from (34a), we have that

(35) $\qquad \delta_k = \lambda \sum_1^m v_{kj}(\alpha_j - R), \quad k = 1, \ldots, m,$

and from (34b) and (35), that

(36) $\qquad \alpha = R + \lambda \sum_1^m \sum_1^m v_{ij}(\alpha_i - R)(\alpha_j - R)$

$\qquad\qquad = R + \lambda[CR^2 - 2AR + B].$

Multiplying (34a) by δ_i and summing from one to m, we have that

(37) $\qquad \lambda = \sum_1^m \sum_1^m \delta_i \delta_j \sigma_{ij}/(\alpha - R)$

$\qquad\qquad = \sigma^2/(\alpha - R).$

By combining (36) and (37) to eliminate λ, the equation for the
frontier can be written as

$$(38) \qquad |\alpha - R| = \sigma \sqrt{CR^2 - 2AR + B} \;,$$

which is drawn in Figure 4. From (35), (37), and (38), the proportions of risky assets for the frontier portfolios as a function of α are

$$(39) \qquad \delta_k = \frac{(\alpha - R) \sum_1^m v_{kj}(\alpha_j - R)}{CR^2 - 2AR + B} \;, \quad k = 1, \ldots, m.$$

As pictured in Figure 4, the frontier is convex (although not strictly convex), and the efficient locus is that portion of the frontier where $\alpha \geq R$. Since the efficient locus is linear in σ, all efficient portfolios are perfectly correlated. From (38) and (39), the lower (inefficient) part of the frontier represent short sales of the risky holdings of the efficient portfolio with the same σ.

Because all efficient portfolios are perfectly correlated, it is straightforward to show that theorem I. holds in the case then one of the securities is risk-less, by simply selecting any two distinct portfolios on the frontier. However, one usually wants a theorem stronger than theorem I when one of the assets is risk-less: namely, that the two mutual funds can be chosen such that one fund holds only the risk-less security and the other fund contains only risky assets (i.e., in the notation of the previous section, $a_{m+1} = 0$ and $b_k = 0$, $k = 1$, . . ., m).

Theorem II. Given m assets satisfying the conditions of section II and a risk-less asset with return R, there

Figure 4.
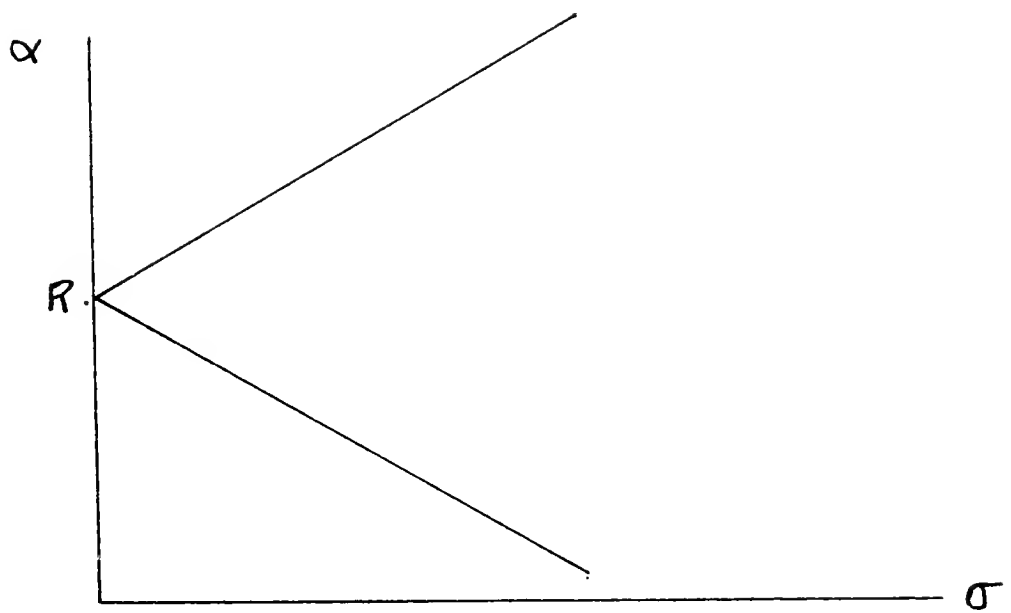
exists a unique pair of efficient[10] mutual funds, one con-
taining only risky assets and the other only the risk-less
asset, such that all risk-averse individuals, who choose
their portfolios so as to maximize utility functions dependent
only on the mean and variance of their portfolios, will be
indifferent between choosing portfolios from among the ori-
ginal m+1 assets or from these two funds, if and only if

$R < \bar{\alpha}$.

The proof of theorem II follows the approach to proving theorem I. If

$$u_k \equiv \Sigma'^m_1 \, v_{kj}(\alpha_j - R)/(CR^2 - 2AR + B), \text{ then}$$

$$\Sigma^m_1 \, u_k = (A - RC)/(CR - 2AR + B).$$

Define $\lambda = \nu(\alpha - R) + \eta, \, (\nu \neq 0)$, then we have that

(40) $\qquad \delta_k = (\alpha - R) \, u_k = \lambda \, a_k + (1 - \lambda) \, b_k$

$$= \nu(\alpha - R)(a_k - b_k) + (1 - \eta)(a_k - b_k) + b_k$$

$$k = 1, \ldots, m,$$

and

(41) $\qquad \delta_{m+1} = 1 - (\alpha - R)(A - RC)/(CR^2 - 2AR + B)$

$$= \nu(\alpha - R)(a_{m+1} - b_{m+1}) + (1 - \eta)(a_{m+1} - b_{m+1}) + b_{m+1}$$

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[10]As mentioned in footnote 9, a mutual fund which is never held long by any investor (which would be the case with risky portfolios along the inefficient part of the frontier) violates the spirit, if not the mathematics, of the mutual fund theorem.

where $a_{m+1} = 1 - \sum_1^m a_k$ and $b_{m+1} = 1 - \sum_1^m b_k$. Solving (40) for a_k and b_k, we have that

(42)
$$a_k = \eta \, u_k / \nu$$

$$b_k = (\eta - 1) \, u_k / \nu \quad , \quad k = 1, \ldots, m$$

and

(43)
$$a_{m+1} = b_{m+1} - (A - RC) / \nu (CR^2 - 2AR + B)$$

$$b_{m+1} = 1 - (\eta - 1)(A - RC) / \nu (CR^2 - 2AR + B).$$

Now require that one of the funds (say the one with proportions b) hold only the risk-less asset (i.e., $b_k = 0$, $k = 1, \ldots, m$ and $b_{m+1} = 1$) which is accomplished by choosing $\eta = 1$. If it is also required that the other fund hold only risky assets (i.e., $a_{m+1} = 0$), then from (43), $\nu = (A - RC)/(CR^2 - 2AR + B)$. Note that if $R = A/C$, $\nu = 0$ which is not allowed, and as can be seen in (43), in this case, $a_{m+1} = b_{m+1} = 1$. From (42), the two mutual funds are different since $b_k = 0$ for all $k = 1, \ldots, m$ and $a_k \neq 0$ for some k. However $\sum_1^m a_k = 0$, which means that the "risky" fund holds a hedged portfolio of long and short positions whose net value is zero. If $R > A/C$, then $\nu < 0$, and the portfolio is inefficient (i.e. $\alpha_a < R$). If $R < A/C$, then $\nu > 0$, and the portfolio is efficient. When $R < A/C$, the composition of the efficient risky portfolio is

(44)
$$a_k = \frac{\sum_1'^m v_{kj}(\alpha_j - R)}{(A - RC)} \quad , \; k = 1, \ldots, m.$$

Thus, theorem II is proved.

The traditional approach to finding the efficient frontier when one of the assets is risk-less is to graph the efficient frontier for risky assets only, and then to draw a line from the intercept tangent to the efficient frontier as illustrated in Figure 5. Suppose that the point (α^*, σ^*) as drawn in Figure 5 exists. Then one could choose one mutual fund to be the risk-less asset and the other to be (α^*, σ^*) which contains only risky assets by virtue of the fact that (α^*, σ^*) is on the efficient frontier for risky assets only. But, by theorem II, two such mutual funds exist if and only if $R < \bar{\alpha} = A/C$ (as is the case in Figure 5). Analytically, the portfolio with expected return and standard deviation, α^* and σ^*, was derived in equations (31) and (32), and the proportions are identical to those in (44)(as they should be).

The proper graphical solutions when $R \geq \bar{\alpha}$ are displayed in Figures 6 and 7. When $R = \bar{\alpha}$, there is no tangency for finite α and σ, and the frontier lines (with the risk-less asset included) are the asymptotes of the frontier curve for risky assets only. When $R > \bar{\alpha}$, there is a lower tangency and the efficient frontier lies above the upper asymptote. Under no condition can one construct the entire frontier (with the risk-less security included) by drawing tangent lines to the upper and lower parts of the frontier for risky assets
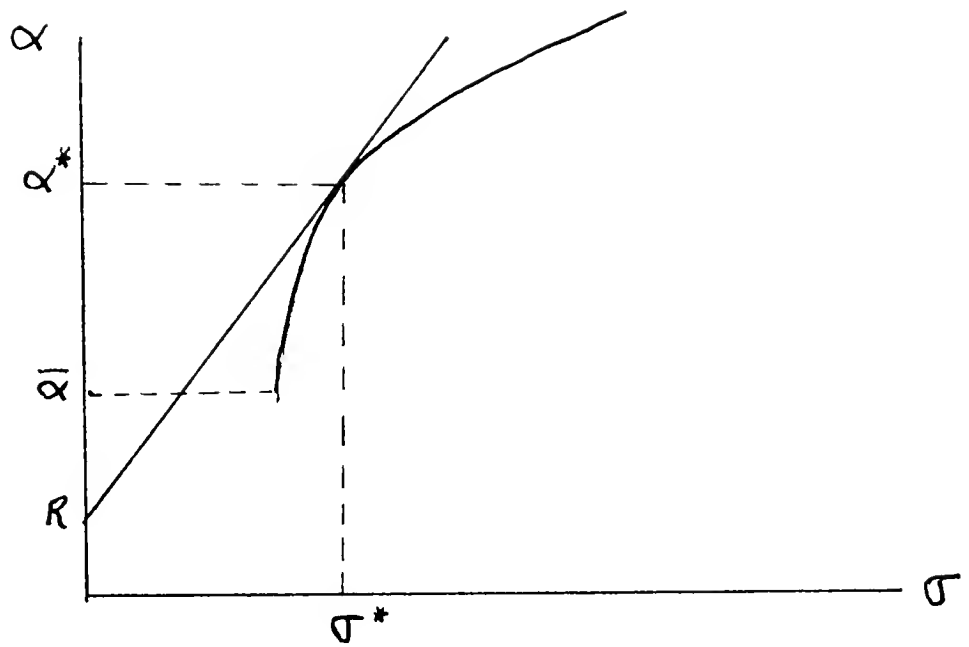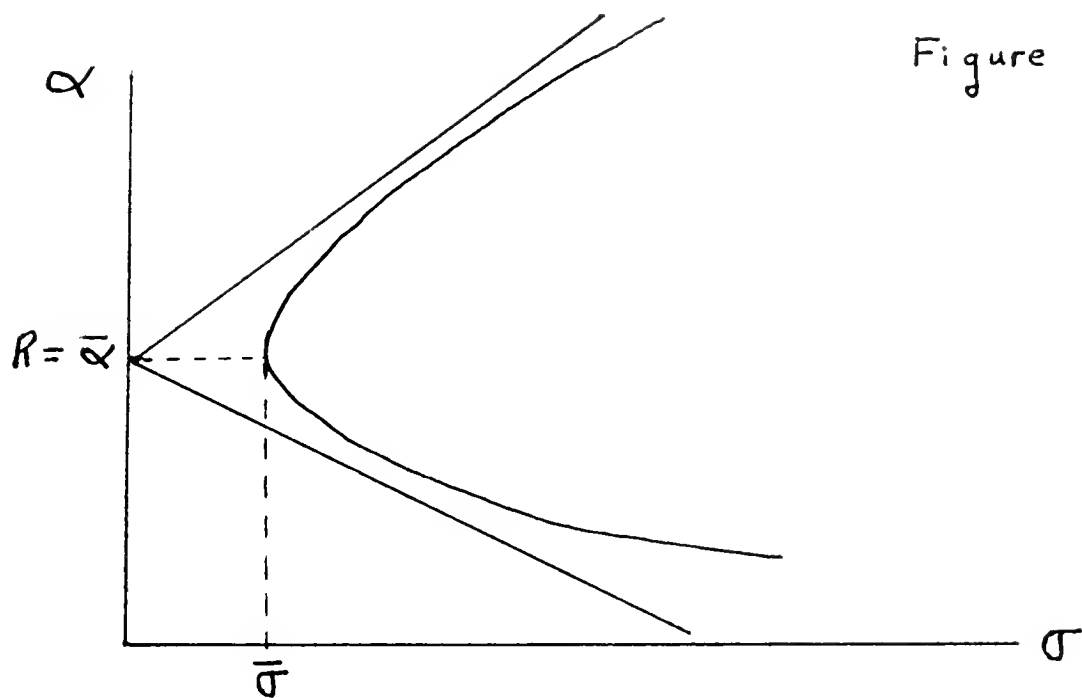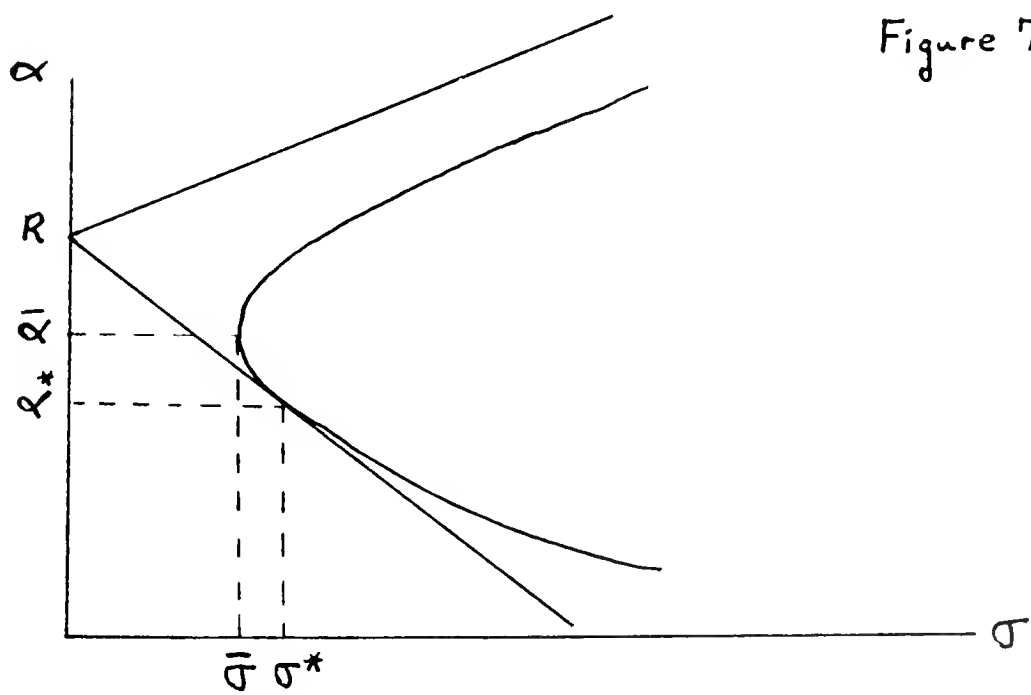
Figure 5.

Figure 6.

Figure 7.

only.[11] The intuitive explanation for this result is that with the introduction of a risk-less asset, it is possible to select a portfolio with net non-positive amounts of risky-assets which was not possible when one could only choose among risky assets.

Although for individual portfolio selection, there is no reason to rule out $R \geq \bar{\alpha}$, one could easily show that as a general equilibrium solution with homogeneous expectations, Figure 5 is the only possible case with (α^*, σ^*), the market portfolio's expected return and standard deviation. Hence, we have as a necessary condition for equilibrium that $R < \bar{\alpha}$.

Given that the proportions in the market portfolio must be the same as in (44)(i.e. $\delta_k^M = a_k$, $k = 1, \ldots$, m where "M" denotes "for the market portfolio"), the fundamental result of the capital asset pricing model, the security market line, can be derived directly as follows:

$$(45) \qquad \sigma_{kM} = \sum_1^m \delta_i^M \sigma_{ij}, \quad k = 1, \ldots, m$$

$$= \sum_1^m \left(\sum_1^m v_{ij}(\alpha_j - R) \right) \sigma_{ik} \bigg/ (A - RC), \text{ from (44)}$$

$$= \sum_1^m (\alpha_j - R) \sum_1^m v_{ij} \sigma_{ik} \bigg/ (A - RC)$$

$$= (\alpha_k - R) \bigg/ (A - RC)$$

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[11]There seems to be a tendency in the literature to draw graphs with $R \geq \bar{\alpha}$ and an upper tangency (e.g. E. Fama [3], p. 26 and M. Jensen [4], p. 174). In W. Sharpe [8], Chapter 4, the figures appear to have $R = \bar{\alpha}$ and a double tangency.

and

(46)
$$\sigma_M^2 \equiv \sum\nolimits_1'^m \mathcal{S}_i^M \sigma_{iM}$$

$$= \sum\nolimits_1'^m \mathcal{S}_i^M (\alpha_i - R)/(A - RC), \text{ from (45)}$$

$$= (\alpha_M - R)/(A - RC)$$

and eliminating (A - RC) by combining (45) and (46), we derive

(47)
$$\alpha_k - R = \frac{\sigma_{kM}}{\sigma_M^2} (\alpha_M - R) , \quad k = 1, \ldots, m,$$

which is the security market line.

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